Order Statistics

(1)

Joint distribution of order statistics of $U(0,1)$.

$$\mathbb{P}(X_{(i)} \in du, X_{(j)} \in dv) \qquad \underset{i-1 \quad 1 \quad j-i-1 \quad 1 \quad n-j}{\rule{3cm}{0.4pt}}$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} u^{i-1} \cdot du \cdot (v-u)^{j-i-1} \cdot dv \cdot (1-v)^{n-j}$$

Hence, the density is

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!\, u^{i-1}(v-u)^{j-i-1}(1-v)^{n-j}}{(i-1)!\,(j-i-1)!\,(n-j)!}$$

(2) Distribution of $\underset{u}{X_{(k)}} - \underset{v}{X_{(j)}}$, $\quad n \geq k > j \geq 1$

Let $z = v - u \qquad 1 \geq v = z + u \geq 0 \Rightarrow 1-z \geq u \geq -z$

$$f(z) = \int_0^1 f(u, u+z)\, du$$

$$= \int_0^{1-z} \frac{\Gamma(n+1)\, u^{i-1} z^{j-i-1} (1-z-u)^{n-j}}{\Gamma(i)\Gamma(j-i)\Gamma(n-j+1)} \cdot \frac{\Gamma(n-(j-i)+1)}{\Gamma(n-(j-i)+1)} du$$

$$= \frac{z^{j-i-1}(1-z)^{n-(j-i)}}{B(j-i,\, n-(j-i)+1)} \cdot \int_0^{1-z} \frac{\left(\frac{u}{1-z}\right)^{i-1}\left(1-\frac{u}{1-z}\right)^{n-j}}{B(i, n-j+1)}\, d\frac{u}{1-z}$$

$$= \mathrm{Beta}(j-i,\, n-(j-i)+1)(z)$$

Specifically, $X_{(n)} - X_{(1)} \sim \mathrm{Beta}(n-1, 2)$

(3) Distribution of $X_{(k)}$. ($X_i \sim U(0,1)$)

$$\mathbb{P}(X_{(k)} \in du) = \frac{n!\, u^{k-1} \cdot du \cdot (1-u)^{n-k}}{(k-1)!\,(n-k)!}$$

$$\Rightarrow f_{X_{(k)}}(u) = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} u^{k-1}(1-u)^{n-k}$$

$$\Rightarrow X_k \sim Beta(k, n-k+1)$$

Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Commonly used integral result

$$\int e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}} \cdot e^{\frac{b^2}{4a}}$$

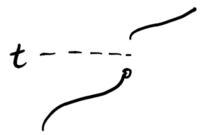
$$\int_a^b x e^{-\frac{x^2}{2}} dx = \int_a^b d(-e^{-\frac{x^2}{2}})$$

$$\int_a^b e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}(\Phi(b) - \Phi(a))$$

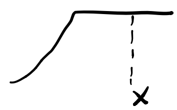
F^{-1}

① Definition : $F^{-1}(t) = \inf\{x : F(x) \geq t\}$

② special examples



$F(F^{-1}(t)) > t$



$F^{-1}(F(x)) < x$

③ For any $t \in [0,1]$, $\mathbb{P}(F(X) \leq t) \leq t$, with equality iff t lies in the closure of the range of F : $\overline{F(-\infty, +\infty)}$

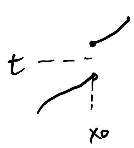
$F(-\infty, +\infty) := \{t : \exists x \in (-\infty, +\infty), \text{ s.t. } F(x) = t\}$

Proof:

(1) If $\exists x_0$ s.t. $F(x_0) = t$, then

$\mathbb{P}(F(X) \leq t) = \mu\{x : F(x) \leq t\} = \mu((-\infty, x_0]) = F(x_0) = t$

(2) If $\nexists x_0$ s.t. $F(x_0) = t$, then define x_0 to be



$F(x_0^-) \leq t < F(x_0^+) = F(x_0)$, then

$\mathbb{P}(F(X) \leq t) = \mu((-\infty, x_0)) = F(x_0^-) \leq t$, when

$t = F(x_0^-)$ equality holds.

④ If F is continuous, then $\mathbb{P}(F(X) \leq t) = t$, then

$F(X) \sim \text{Uniform}(0,1)$

Poisson

1. $X \sim \text{Poisson}(\theta)$, show $Var(X) = \theta$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \cdot \frac{\theta^x}{x!} e^{-\theta} = \theta \sum_{x=1}^{\infty} \frac{\theta^{x-1}}{(x-1)!} e^{-\theta} = \theta$$

$$\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X]$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{\theta^x}{x!} e^{-\theta} + \theta$$

$$= \theta^2 \sum_{x=2}^{\infty} \frac{\theta^{x-2}}{(x-2)!} e^{-\theta} + \theta = \theta^2 + \theta$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \theta$$

2. Poisson is DQM, so in the i.i.d case is LAN.

Calculation Tricks

1. $$\beta'(\theta) = \frac{d}{d\theta}\int \varphi p_\theta \, d\mu = \int \varphi \frac{\partial p_\theta}{\partial \theta} \, d\mu = \int \varphi \frac{\partial \log p_\theta}{\partial \theta} \cdot p_\theta \, d\mu$$
$$= \mathbb{E}_\theta \varphi \, l'(\theta)$$

2. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ancillary statistic in Normal distribution

(1) $X_1, \cdots, X_n \sim N(\mu, \sigma_0^2)$

$$V = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

(2) $X_1, \cdots, X_n \sim N(\mu_0, \sigma^2)$

$$V = \frac{\bar{X} - \mu_0}{\sqrt{\sum_{i=1}^{n} (X_i - \mu_0)^2}} \quad \text{or} \quad V = \frac{\bar{X} - \mu_0}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2}}$$

(3) $X_1, \cdots, X_n \sim N(\xi, \sigma^2)$, $Y_1, \cdots, Y_n \sim N(\eta, \tau^2)$

$$T = (\bar{X}, \sum_i X_i^2, \bar{Y}, \sum_i Y_i^2)$$

$$V = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_i (X_i - \bar{X})^2 (Y_i - \bar{Y})^2}}$$

UMPU test problem IV

$$\varphi(T) = \begin{cases} 1 & T > C_2 \text{ or } T < C_1 \\ \gamma_1 & T = C_1 \\ \gamma_2 & T = C_2 \\ 0 & C_1 < T < C_2 \end{cases}$$

s.t. $\mathbb{E}_{\theta_0} \varphi(T) = \alpha$

$$\mathbb{E}_{\theta_0} T \varphi(T) = \alpha \mathbb{E}_{\theta_0} T$$

when the distribution of T is symmetric about some point, it is easy to solve the equations.

First, note that if X is symmetric about a, then $\mathbb{E} X = a$; if $f(x)$ is symmetric about a, $\mathbb{E} f(x) h(x) = a$, then $h(x)$ is symmetric about a.

Now, let's derive the solution.

$$\mathbb{E}_{\theta_0} T \varphi(T) = \alpha \mathbb{E}_{\theta_0} T = \alpha a = a \mathbb{E}_{\theta_0} \varphi(T)$$

$$\Rightarrow \mathbb{E}_{\theta_0} (T-a) \varphi(T) = 0$$

$$= \int \underset{\substack{\uparrow \\ \text{symm. about } 0}}{(t-a)} \underset{\substack{\Downarrow \\ a}}{\varphi(t)} \underset{\substack{\uparrow \\ a}}{f_{\theta_0}(t)} \, dt = 0$$

$\Rightarrow \varphi(T)$ is symmetric about a.

Then we have $\quad c_1 + c_2 = 2a$

$$\gamma_1 = \gamma_2$$

$$\mathbb{P}_{\theta_0}(T < c_1) + \gamma_1 \mathbb{P}_{\theta_0}(T = c_1) = \alpha$$

Beta distribution

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} , \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$